Worksheet 4 Solutions

MATH 33A

1. We start by applying the Gram-Schmidt process to the given basis vectors of $\mathbb{R}^2$:

   First, let's define the vectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix},$$
$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

   The Gram-Schmidt process starts by setting $\mathbf{u}_1 = \mathbf{v}_1$:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

   Next, we subtract from $\mathbf{v}_2$ its projection onto $\mathbf{u}_1$:

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \frac{14}{10}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ -\frac{1}{5} \end{bmatrix}.$$

   Now we normalize $\mathbf{u}_1$ and $\mathbf{u}_2$ to obtain the orthonormal basis:

$$\mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
$$\mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

   So, the orthonormal basis for $\mathbb{R}^2$ is $\{\mathbf{e}_1, \mathbf{e}_2\}$.

   Now, to find the QR decomposition of the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, we first form matrix Q using our orthonormal vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ as columns:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{bmatrix}.$$

Next, we calculate the matrix R by calculating the values $\mathbf{v}_1 \cdot \mathbf{e}_1 = \sqrt{10}$, $\mathbf{v}_2 \cdot \mathbf{e}_1 = \frac{14}{\sqrt{10}}$, $\mathbf{v}_2 \cdot \mathbf{e}_2 = \frac{2}{\sqrt{10}}$ and putting them above the diagonal.

So, the QR decomposition of the given matrix is

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & -\frac{1}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} \sqrt{10} & \frac{14}{\sqrt{10}} \\ 0 & \frac{2}{\sqrt{10}} \end{bmatrix}.$$

2. An isomorphism is a bijective (one-to-one and onto) linear transformation. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ and $T' : \mathbb{R}^n \to \mathbb{R}^n$ be isomorphisms. To show that the composition $T \circ T'$ is also an isomorphism, we need to show that it is linear and bijective.

   Linearity: Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} (T \circ T')(\alpha\mathbf{u} + \mathbf{v}) &= T(T'(\alpha\mathbf{u} + \mathbf{v})) \\ &= T(\alpha T'(\mathbf{u}) + T'(\mathbf{v})) \\ &= \alpha T(T'(\mathbf{u})) + T(T'(\mathbf{v})) \\ &= \alpha(T \circ T')(\mathbf{u}) + (T \circ T')(\mathbf{v}), \end{aligned}$$

   so $T \circ T'$ is linear.

   Bijectivity: Since T and T' are both bijective, they have inverses T^{-1} and T'^{-1}. The inverse of $T \circ T'$ is $T'^{-1} \circ T^{-1}$, which is well-defined, showing that $T \circ T'$ is bijective. Therefore, $T \circ T'$ is an isomorphism.

3. The volume of the unit sphere in $\mathbb{R}^3$ is $\frac{4}{3}\pi$. The linear transformation $T : (x, y, z) \to (ax, by, cz)$ maps the unit sphere to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

   The determinant of the matrix representing T is abc, which is the factor by which T scales volumes. Hence, the volume of the ellipsoid is $\frac{4}{3}\pi abc$.

4. If λ is an eigenvalue of A with eigenvector v, where A^3 is the identity matrix, then $A^3 v = \lambda^3 v = v$, implying that $\lambda^3 = 1$. Thus, the possible eigenvalues of A are the roots of $\lambda^3 - 1 = 0$, which are $1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}$.