Worksheet 4 Solutions

MATH 33A

1. We are given that the subspace W is spanned by the plane $x+2y+z=0$ in $\mathbb{R}^3$. A plane in $\mathbb{R}^3$ is a two-dimensional object, and thus $\dim W = 2$.

   Now, we need to find a basis for W. To do this, we must find two vectors that are not collinear and lie on the plane. This can be achieved by finding two distinct solutions to the plane equation, which will give us two vectors in the plane.

   Rearranging the equation for x, we have $x = -2y - z$.

   Choosing $y = 1, z = 0$ yields $x = -2$, and thus we have a vector $\mathbf{v}_1 = (-2, 1, 0)$.

   Next, choosing $y = 0, z = 1$ yields $x = -1$, and thus we have a vector $\mathbf{v}_2 = (-1, 0, 1)$.

   Therefore, a basis for W is $\{\mathbf{v}_1, \mathbf{v}_2\} = \{(-2, 1, 0), (-1, 0, 1)\}$.

   The number of elements in the basis is equal to the dimension of the subspace, so there are 2 elements in the basis of W.

2. The linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 3 & 0 \end{bmatrix}$.

   - **Image of** T: The image of T, denoted by $\text{im}(T)$, is the set of all vectors that can be reached by applying the transformation T to vectors in the domain. It is also equal to the column space of the matrix A. Here, the image of T can be spanned by the column vectors of A, which are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$. So, $\text{im}(T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\} = \mathbb{R}^3$ (since any vector can be written as a linear combination of these three column vectors).
   - **Rank of** A: The rank of a matrix A, denoted by $\text{rank}(A)$, is the dimension of the image of A. Since $\text{im}(A) = \mathbb{R}^3, \text{rank}(A) = 3$.

- The columns of A are linearly independent because the dimension of the their span (i.e. the rank of A) is is equal to the number of columns. Therefore, they form a basis for $\mathbb{R}^3$, implying that they indeed span $\mathbb{R}^3$.
- **Kernel of** T: The kernel of T, denoted by $\ker(T)$, is the set of all vectors v in the domain such that $T(v) = 0$. It is also equal to the null space of the matrix A. By the rank-nullity theorem, $\dim \ker T = 3 - \operatorname{rank}(A) = 0$. Hence, $\ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

3. We are given two bases for $\mathbb{R}^n$, $X = \{e_1, \ldots, e_n\}$ and $Y = \{w_1, \ldots, w_n\}$, and a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. We want to show that $T_Y = PT_X P^{-1}$.

   Let $A = [T]_X$ and $B = [T]_Y$ be the matrix representations of T with respect to the bases X and Y, respectively. The goal is to show that $B = PAP^{-1}$.

   By definition, for any vector v in $\mathbb{R}^n$, we have $Av = T(v)$ when v is represented in the basis X, and $Bv = T(v)$ when v is represented in the basis Y.

   The matrix P is the change of basis matrix from X to Y. Therefore, if v_X is the representation of v in the basis X, then $v_Y = Pv_X$ is the representation of v in the basis Y. We also know that $v_X = P^{-1}v_Y$.

   Substituting this into the equation for Av gives $AP^{-1}v_Y = T(v)$.

   Now, let's look at BPv. Since Pv gives us the coordinates of v in the basis Y, we have $BPv = T(v)$.

   Therefore, we have $AP^{-1}v_Y = BPv$, which implies $BP = AP^{-1}$ and consequently $B = PAP^{-1}$.

   Hence, we have shown that the matrix of a linear transformation changes according to the formula $T_B = PT_A P^{-1}$ when we change the basis.

4. To find an orthonormal basis for $\operatorname{im}(A)$, we can apply the Gram-Schmidt process to the column vectors of A.

   First, let's define the column vectors of A:

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 3 \end{bmatrix},$$

$$\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix},$$

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix},$$

$$\mathbf{v}_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix}.$$

We start by setting $\mathbf{u}_1 = \mathbf{v}_1$:

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 3 \end{bmatrix}.$$

Next, we subtract from $\mathbf{v}_2$ its projection onto $\mathbf{u}_1$:

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_2}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{14}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{22}{7} \\ \frac{13}{14} \\ \frac{25}{14} \end{bmatrix}.$$

Similarly, we find $\mathbf{u}_3$ and $\mathbf{u}_4$:

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_3}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_3}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} - \frac{-7}{14}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 3 \end{bmatrix} - \frac{-49}{209}\begin{bmatrix} 1 \\ \frac{22}{7} \\ \frac{13}{14} \\ \frac{25}{14} \end{bmatrix} = \begin{bmatrix} \frac{258}{209} \\ -\frac{5}{19} \\ -\frac{59}{209} \\ -\frac{17}{209} \end{bmatrix},$$

$$\mathbf{u}_4 = \mathbf{v}_4 - \frac{\mathbf{u}_1 \cdot \mathbf{v}_4}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_4}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_4}{\mathbf{u}_3 \cdot \mathbf{u}_3}\mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \frac{8}{14}\begin{bmatrix} 0 \\ -2 \\ 1 \\ 3 \end{bmatrix} - \frac{160}{209}\begin{bmatrix} 1 \\ \frac{22}{7} \\ \frac{13}{14} \\ \frac{25}{14} \end{bmatrix} - \begin{bmatrix} \frac{258}{209} \\ -\frac{5}{19} \\ -\frac{59}{209} \\ -\frac{17}{209} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $\mathbf{u}_4$ is the zero vectors, it does not contribute to the basis.

Finally, we normalize $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to obtain the orthonormal basis:

$$\mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} = \frac{1}{\sqrt{14}} \begin{bmatrix} 0 \\ -2 \\ 1 \\ 3 \end{bmatrix},$$
$$\mathbf{e}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \frac{14}{\sqrt{2926}} \begin{bmatrix} 1 \\ \frac{22}{7} \\ \frac{13}{14} \\ \frac{25}{14} \end{bmatrix}$$
$$\mathbf{e}_3 = \frac{\mathbf{u}_3}{\|\mathbf{u}_3\|} = \frac{209}{3\sqrt{8151}} \begin{bmatrix} \frac{258}{209} \\ -\frac{5}{19} \\ \frac{59}{209} \\ -\frac{17}{209} \end{bmatrix}.$$

Therefore, the orthonormal basis for $\operatorname{im}(A)$ is $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.