Worksheet 3

MATH 33A

1. Show that the family of linear transformations $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$A_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

satisfy the following properties:

   (a) They commute, i.e. $A_\theta A_{\theta'} = A_{\theta'} A_\theta$.

   (b) They are 2π-periodic, i.e. $A_{\theta+2\pi} = A_\theta$.

   (c) They rotate the unit vector e_1 by θ degrees to the vector $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$.

2. (a) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, find A^{-1}. Show that $AA^{-1} = A^{-1}A = I_2$. Does every matrix have an inverse? What is the condition for a 2×2 matrix to be invertible?

   (b) Show that if x, b are $n \times 1$ column vectors, A is an $n \times n$ matrix, $Ax = b$, and A is invertible, then x is given by $x = A^{-1}b$. If A is as above, solve the system $Ax = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

   (c) Show that $(AB)^{-1} = B^{-1}A^{-1}$.

   (d) Suppose A is an $n \times n$ matrix such that all the values a in A satisfy $|a| \leq r < \frac{1}{n}$. For some positive number $r < 1$. Show that all values in A^k are bounded by r for any k. Show that under these conditions, $I_n - A$ is invertible and

$$(I_n - A)^{-1} = I + A + A^2 + ...$$

3. Find a linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ mapping the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and the vector $\begin{bmatrix} -2 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

4. Show that $W \subset \mathbb{R}^3 = \{(x, y, z) : x + y + z = 0\}$ is a linear subspace. Find a set of linearly independent vectors that span W. What is the dimension of W?