Worksheet 1 Solutions

MATH 33A

1. (a) To find $A+B, AB, BA, A^2$, and B^2, perform the following calculations:

$$A+B = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 3 & 3 \end{bmatrix}.$$

$$AB = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ -2 & 1 & 1 \\ 8 & 3 & 4 \end{bmatrix}.$$

$$BA = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 6 \\ 2 & 3 & 3 \\ 3 & 5 & 2 \end{bmatrix}.$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ -2 & 1 & -2 \\ 2 & 4 & 5 \end{bmatrix}.$$

$$B^2 = B \cdot B = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 9 & 0 & 0 \\ 4 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}.$$

Now, let's check if $(A + B)^2 = A^2 + AB + BA + B^2$:

$$(A + B)^2 = \left(\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 3 & 3 \end{bmatrix} \right)^2$$
$$= \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 3 & 3 \end{bmatrix} \cdot \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 3 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} 20 & 6 & 14 \\ 2 & 7 & 5 \\ 14 & 15 & 16 \end{bmatrix}.$$

$$A^2 + AB + BA + B^2 = \begin{bmatrix} 5 & 4 & 4 \\ -2 & 1 & -2 \\ 2 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 4 \\ -2 & 1 & 1 \\ 8 & 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 6 \\ 2 & 3 & 3 \\ 3 & 5 & 2 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 4 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$$
$$= \begin{bmatrix} 20 & 6 & 14 \\ 2 & 7 & 5 \\ 14 & 15 & 16 \end{bmatrix}.$$

Since $(A + B)^2 = A^2 + AB + BA + B^2$, the equation holds true. However, $(A+B)^2 \neq A^2 + 2AB + B^2$, as matrix multiplication is not commutative.

(b) To show that $A \cdot I_n = I_n \cdot A = A$ for any $n \times n$ matrix A, consider the following:

Let A be an arbitrary $n \times n$ matrix, and let I_n be the $n \times n$ identity matrix.

$$A \cdot I_n = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Each element of the resulting matrix can be computed as follows:

$$
\begin{aligned}
(A \cdot I_n)_{ij} &= \sum_{k=1}^{n} a_{ik} \cdot \delta_{kj} \\
&= a_{i1} \cdot \delta_{1j} + a_{i2} \cdot \delta_{2j} + \cdots + a_{in} \cdot \delta_{nj}.
\end{aligned}
$$

Notice that $\delta_{kj} = 1$ only if $k = j$ and $\delta_{kj} = 0$ otherwise. Thus, the above expression simplifies to:

$$(A \cdot I_n)_{ij} = aij,$$

which means that $A \cdot I_n = A$. Similarly, one can show that $I_n \cdot A = A$. Therefore, for any $n \times n$ matrix A, we have $A \cdot I_n = I_n \cdot A = A$.

(c) Suppose P is a 2×3 matrix and Q is a 3×4 matrix. To determine if the products PQ and QP exist, recall that the product of two matrices is defined only if the number of columns in the first matrix is equal to the number of rows in the second matrix.

For PQ, since P has 3 columns and Q has 3 rows, the product PQ exists. The resulting product matrix will have the same number of rows as P and the same number of columns as Q, which is 2×4.

For QP, since Q has 4 columns and P has 2 rows, the product QP does not exist, as the number of columns in Q does not match the number of rows in P.

2. We are given the linear system of equations:

$$
\begin{aligned}
2x + 3y &= 3, \\
x + 2y &= 3.
\end{aligned}
$$

Let's use the substitution method to solve this system. From the second equation, we can solve for x:

$$x = 3 - 2y.$$

Now substitute this expression for x into the first equation:

$$
\begin{aligned}
2(3 - 2y) + 3y &= 3 \\
6 - 4y + 3y &= 3 \\
y &= 3.
\end{aligned}
$$

Now that we have the value of y, we can substitute it back into the expression for x:

$$
\begin{aligned}
x &= 3 - 2(3) \\
x &= -3.
\end{aligned}
$$

Thus, the solution to the given system of equations is $x = -3$ and $y = 3$.

3. First, let's solve the system using Gaussian elimination. We start with the augmented matrix and perform the following Gaussian operations:

$$\left[\begin{array}{cc|c} 2 & 3 & 3 \\ 1 & 2 & 3 \end{array}\right] \xrightarrow{R_1 = \frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{3}{2} \\ 1 & 2 & 3 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{3}{2} \end{array}\right] \xrightarrow{R_2 = 2R_2} \left[\begin{array}{cc|c} 1 & \frac{3}{2} & \frac{3}{2} \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1 = R_1 - \frac{3}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 3 \end{array}\right]$$

Thus, the solution to the system is the same as before: $x = -3$ and $y = 3$.

4. (a) Solve the system

$$\begin{aligned} 2x + 3y + z &= 3, \\ x + 2y + z &= 3, \end{aligned}$$

using Gaussian elimination.
The augmented matrix for this system is:

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 3 \\ 1 & 2 & 1 & 3 \end{array}\right] \xrightarrow{R_1 = \frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\ 1 & 2 & 1 & 3 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{array}\right]$$
$$\xrightarrow{R_2 = 2R_2} \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow{R_1 = R_1 - \frac{3}{2}R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

The reduced row echelon form of the matrix corresponds to the system of equations:

$$\begin{aligned} x - z &= -3, \\ y + z &= 3. \end{aligned}$$

This system is underdetermined, as there are more variables than equations. The set of solutions can be described as the line $\langle z-3, 3-z, z\rangle = \langle 1, -1, 1\rangle z + \langle -3, 3, 0\rangle$ in the xyz-plane, where any point on the line (x, y, z) satisfies the given system of equations.

(b) Solve the system

$$\begin{aligned} x + 2y + z &= 1, \\ x + 3y + 2z &= 3, \\ y + z &= 3, \end{aligned}$$

using Gaussian elimination. The augmented matrix for this system is:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The reduced row echelon form of the matrix corresponds to the following system of equations:

$$\begin{aligned} x + 2y + z &= 1, \\ y + z &= 2, \\ 0 &= 1. \end{aligned}$$

The last equation is a contradiction, which means this system has no solution.

(c) Based on the examples above, the reduced row echelon form of a matrix tells us about the set of solutions to a system of equations. If the reduced row echelon form corresponds to a consistent system of equations (i.e., no contradictions), then the system has a solution. If the system is underdetermined, the solution set will contain an infinite number of solutions, and if the system is overdetermined, the solution set can either be empty (as in the second example) or have a unique solution (as in the first example).