Worksheet 1

MATH 33A

1. Let
$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

   (a) Find $A + B, AB, BA, A^2, B^2$. Show that
   $$(A + B)^2 = A^2 + AB + BA + B^2.$$
   Is it true that
   $$(A + B)^2 = A^2 + 2AB + B^2?$$
   (Note: the above exercise shows that matrix multiplication is not necessarily commutative, i.e. it is not necessarily true that $AB = BA$ for any two matrices A, B.)

   (b) Define the $n \times n$ identity matrix I_n to be
   $$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$
   Show that for any $n \times n$ matrix A, $A \cdot I_n = I_n \cdot A = A$.

   (c) Suppose P is a 2×3 matrix and Q is a 3×4 matrix. Does the product PQ exist? What about QP? If so, what is the size of each product matrix?

2. Solve the linear system of equations
$$\begin{aligned} 2x + 3y &= 3 \\ x + 2y &= 3 \end{aligned}$$
using any methods you know so far.

3. Now, let's solve the same problem using Gaussian elimination. One begins by writing the augmented matrix of coefficients

$$\left[\begin{array}{cc|c} 2 & 3 & 3 \\ 1 & 2 & 3 \end{array}\right]$$

and using the three Gaussian operations (switch two rows, adding a multiple of a row to another row, and multiplying a row by a nonzero constant, to bring the the matrix to its reduced row echelon form. The right most column will be the set of solutions to the system. Do you get the same result? (Note: the above system has a unique solution, which happens precisely when the reduced row echelon form of the augment matrix is the identity matrix I_n.)

4. (a) Solve the system

$$\begin{aligned} 2x + 3y + z &= 3 \\ x + 2y + z &= 3 \end{aligned}$$

using Gaussian elimination. Describe the set of solutions geometrically. (Note: this is an example of an underdetermined system.)

(b) Solve the system

$$\begin{aligned} x + 2y + z &= 1 \\ x + 3y + 2z &= 3 \\ y + z &= 3 \end{aligned}$$

using Gaussian elimination. (Note: this is an example of an overdetermined system.)

(c) Based on the examples above, what does the reduced row echelon form of a matrix tell us about the set of solutions to a system of equations?